Z-quasi prime submodules and the Z-radicals of submodules

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Abstract. Let R be a commutative ring with identity and N be a proper submodule N of R-moduLe M is called prime if whenever $rx \in N$; $r \in R$, $x \in M$, implies either $x \in N$ or $r \in [N : M]$, where $[N : M] = \{r \in R : rM \subseteq N\}$. In this paper we say that N is Z-qusi prime submodule of R-moduLe M, if $[N : (x)]$ is a Z-prime ideal of R for each $x \in M$. We prove some result of this type of submodules.

Keywords: prime submodule, Z-prime submodule, qusi prime submodule, Z-regular module.

1. Introduction

A proper submodule N of module M over a ring R is said to be prime (or P-prime) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $r \in [N : M]$ or $x \in N$, where $[N : M] = \{r \in R : rM \subseteq N\}$, [1]. And M is called prime module if the zero submodule of M is a prime submodule of M, [2]. In the last year many studies and researches are published about prime submodule by many people who care with the subject of commutative algebra and some of them are J. Dauns, R.L. Mcsland, C.P. Lu, P.F. Smith, M.E. Moore. M is called faithful module if $[0 : M] = 0$,[3]. Where $[0 : M] = \{r \in R : rM = 0\}$ and $[0 : N]$ is define as $[0 : N] = \{r \in R : rN = 0\}$. There are several generalization of the notion of a prime submodules as like S-prime, [4]. N. S. Al-Mothafar and A. T. Husain [5] defined the Z-prime submodule of an R-module Mas follows: A proper submodule N of module M over a ring R is said to be Z-prime if $f(x).x \in N$ for $f \in M^* = Hom(M, R)$, $x \in M$ implies that either $f(x) \in [N : M]$ or $x \in N$ and show that every prime submodule is Z-prime submodule but the converse is not true in general. However they give condition under which a proper Z-prime submoduleis a prime submodule. In this paper we study the concept of a Z-quasi prime submodule of an R-module M, we give some examples and properties and give the relation between Z-prime submodules and Z-quasi prime submodules. Also, we define the notation of $Z - rad(N)$ using the concept of Z-prime submodules and we study some of there basic properties.

2. Z-quasi prime submodule

Definition 2.1. A proper submodule N of an R-module M is said to be Z-quasi prime submodule of M if $[N : (x)]$ is a Z-prime ideal of R for each $x \in M$.

Remarks and Examples 2.2.

1. Every quasi-prime submodule is Z-quasi prime submodule but the converse is not true, for example the submodule Z of the Z-module Q is Z-quasi prime submodule, since Hom $(Q, Z) = 0$, but not quasi-prime submodule.
2. If N is a Z-prime submodule of an R-module M such that $ann(x) = 0$ for each $x \in M$, then $[N : (x)]$ is a Z-prime ideal of R
3. If N is a Z-prime submodule of an R-module M, then N is a Z-quasi prime submodule of M and $ann(x) = 0$; for each $x \in M$.
4. Consider the Z-module $M = Z \oplus Z_p$, where P is a prime number and the Z-submodule $N = qZ \oplus Z_p$, where q is any prime number. Then, for any $x \in M$ implies that $[N : (x)] = qZ$, which is a Z-prime ideal of Z. Therefore N is a Z-quasi prime submodule of M.

Proof. (2) Let $f(a).a \in [N : (x)]$; $a \in R$, $f \in Hom(M, R) = R^*$, $x \in M$ and suppose $f(a) \notin [N : (x)]$, which implies that $f(a).\ a(x) \subseteq N$, $f(a).a(x) \in N$. Define $h : M \to R$ by $h(rx) = r$; for each $r \in R$, then $f(h(ax)).\ ax \in N$, since N is a Z-prime submodule of M and $f \circ h : M \to R$, then either $ax \in N$, which implies that $a \in [N : (x)]$ or $f(h(ax)) = f(a) \in [N : (x)]$ which is a contradiction. □

The following theorem gives some characterizations for Z-quasi prime submodule.

Theorem 2.3. *Let N be a proper submodule of an R-module M, then the following are equivalent:*

1. *N is a Z-quasi prime submodule of M.*
2. *$[N : K]$ is a Z-prime ideal of R for each submodule K of M*
3. *$[N : (rm)] = [N : (m)]$ for each $m \in M$, $r \in R$ and $r \notin [N : (m)]$.*

Proof. (1) ⇒ (2) Let N be a Z-quasi prime submodule of M. Then $[N : (x)]$ is a Z-prime ideal of R, for each $x \in M$. So $[N : (x)]$ is a Z-prime ideal for each $x \in K$ and by lemma (2.1.23) in [5], $[N : K]$ is a Z-prime ideal of R.

(2) ⇒ (3) It is clear that $[N : (m)] \subseteq [N : (rm)]$. Let $a \in [N : (rm)]$ for each $r \notin [N : (m)]$ and $m \in M$. Hence, $a(rm) \subseteq N$. It follows that $ar \in [N : (m)]$. Define $h : R \to R$ by $h(r) = r$, it is clear that $h \in R^* = Hom(R, R)$, $h(a).r \in [N : (m)]$ which is a Z-prime ideal by (2). But $r \notin [N : (m)]$ so $a \in [N : (m)]$. Thus $[N : (rm)] \subseteq [N : (m)]$. Therefore $[N : (m)] = [N : (rm)]$.

(3) $\Rightarrow$ (1) Let $x \in M$ and $f \in Hom(R, R)$, $a \in R$ such that $f(a).a \in [N : (x)]$, suppose $a \notin [N : (x)]$, hence by (3), $[N : (ax)] = [N : (x)]$. But $f(a) \in [N : (ax)]$, so $f(a) \in [N : (x)]$ and hence N is a Z-quasi prime submodule. □

The following is an immediate consequence of above theorem.

Corollary 2.4. *Let N be submodule of an R-module M. If N is a Z-quasi prime submodule of M, then $[N : M]$ is Z-prime ideal of R.*

Recall that an R-module M is said to be Z-regular R-module if for each $x \in M$ there exists $f \in M^* = Hom(M, R)$ such that $f(x).x = x$, [6].

Proposition 2.5. *Let M be a Z-regular R-module, then every Z-quasi prime submodule of M is Z-prime submodule of M.*

Proof. Let $x \in M$, $f \in M^* = Hom(M, R)$ such that $f(x).x \in N$ and $f(x) \notin [N : M]$ we have to show $x \in N$, since M is Z-regular, then for each $x \in M$ there exist $g \in M^*$, such that $g(x).x = x$.

Case 1: If $f = g$.Thus $x = f(x).x \in N$.

Case 2: If $f \neq g$. Thus $f(x).x = f(x)g(x).x$, since N is quasi-Z- prime submodule of M, then either $f(x).x \in N$ or $g(x).x \in N$, if $f(x).x \in N$ we are done, if $g(x).x \in N$, which implies that $x \in N$. Thus N is Z-prime submodule of M. □

The following proposition gives another characterization of Z-quasi prime submodule in case M is Z-regular and multiplication modules.

Proposition 2.6. *Let N be a proper submodule of multiplication, Z-regular R-module M. Then the following statements are equivalent:*

1. *N is a Z-quasi prime submodule.*
2. *$[N : M]$ is a Z-prime ideal .*
3. *N is Z-prime submodule.*

Proof. (1) $\Rightarrow$ (2) , by corollary 2.4.

(2) $\Rightarrow$ (3), by proposition 2.5.

(3) $\Rightarrow$ (1), by remark (2.2) (1). □

Proposition 2.7. *Let M and $\acute{M}$ be two modules and let $f : M \to \acute{M}$ be an epimorphism. If N is a Z-quasi prime submodule of M such that $ker f \subseteq N$, then $f(N)$ is a Z-quasi prime submodule of $\acute{M}$.*

Proof. To prove $f(N)$ is a Z-quasi prime submodule of $\acute{M}$, we must prove $[f(N) : \acute{K}]$ is Z-prime ideal of R, $\forall \acute{K} \supseteq f(N)$. Since f is an epimorphism,then $\acute{K} = f(f^{(-1)}(\acute{K}))$. Let $K = f^{(-1)}(\acute{K})$, so $f(K) = \acute{K}$. It follows that $f(K) \supseteq f(N)$. So to prove $[f(N) : f(K)]$ is a Z-prime ideal of R. Let $h \in Hom(R.R)$, $a \in R$ such that $h(a).a \in [f(N) : f(K)]$ so $h(a)af(K) \subseteq f(N)$, then for each

$x \in K$, $h(a)af(x) \in f(N)$, so that $f(h(a)ax) = f(n)$ for some $n \in N$. Which implies that $h(a)ax - n \in kerf \subseteq N$ and so $h(a)ax \in N$ for each $x \in K$. Hence $h(a).a \in [N : K]$ is Z-prime ideal since N is a Z-quasi prime submodule, so either $h(a) \in [N : K]$ or $a \in [N : K]$. Thus either $h(a)k \subseteq N$ or $aK \subseteq N$ and so either $h(a)f(K) \subseteq f(N)$ or $af(K) \subseteq f(N)$. Therefore either $h(a) \in [f(N) : f(K)]$ or $a \in [f(N) : f(K)]$, which implies that $[f(N) : f(K)]$ is a Z-prime ideal of R and $f(N)$ is a Z-quasi prime submodule of $\acute{M}$. $\square$

Proposition 2.8. *Let M and $\acute{M}$ be two modules and let $f : M \to \acute{M}$ be an epimorphism. If N is a Z-quasi prime submodule of $\acute{M}$, then $f^{(-1)}(N)$ is a Z-quasi prime submodule of M.*

Proof. To prove $f^{(-1)}(N)$ is a Z-quasi prime submodule of M, we must prove $[f^{(-1)}(N) : K]$ is a Z-prime ideal of R, $\forall K \supseteq f^{(-1)}(N)$. Let $h \in Hom(R.R)$, $a \in R$ such that $h(a).a \in [f^{(-1)}(N) : K]$ so $h(a)aK \subseteq f^{(-1)}(N)$. Hence $f(h(a)aK) \subseteq f(f^{(-1)}(N))$, it follows that $h(a)a(f(K)) \subseteq f(f^{(-1)}(N)) = N$. Thus $h(a)a \in [N : f(K)]$. But $K \supseteq f^{(-1)}(N)$ so that $f(K) \supseteq f(f^{(-1)}(N)) = N$ (since f is epimorphism) and hence $[N : f(K)]$ is Z-prime ideal. Hence either $h(a) \in [N : f(K)]$ or $a \in [N : f(K)]$ and so either $h(a)f(K) \subseteq N$ or $a.f(K) \subseteq N$. Therefore either $f(h(a)K) \subseteq N or f(aK) \subseteq N$ that is either $h(a)K \subseteq f^{(-1)}(N)$ or $aK \subseteq f^{(-1)}(N)$. It follows that $h(a) \in [f^{(-1)}(N) : K]$ or $a \in [f^{(-1)}(N) : K]$. Thus $[f^{(-1)}(N) : K]$ is a Z-prime ideal, $\forall K \supseteq f^{(-1)}(N)$ and therefore $f^{(-1)}(N)$ is a Z-quasi prime submodule of $\acute{M}$. $\square$

Corollary 2.9. *Let K and N be two submodules of an R-module and $K \subseteq N$. Then N/K is a Z-quasi prime submodule of M/K if and only if N is a Z-quasi prime submodule of M.*

Proof. Let $\varphi : M \to M/K$ be natural mapping, then the result follows from. proposition 2.7 and proposition 2.8. $\square$

3. The Z-radicals of submodules

The concept of prime radical of a submodules N of of an R-module M, denoted by $rad(N)$, was defined by as the intersection of all prime submodules of M containing N, [7], if N is not contained in any prime submodule, then $rad(N) = M$. In this section, we introduce the notation of Z-rad(N) and we give some of its properties.

Definition 3.1. Let N be a submodule of an R-module. The intersection of all Z-prime submodules of M containing N is called the Z-radical of N and it is denoted by Z-rad(N). If no Z-prime submodule of M containing N, then we put $Z - rad(N) = M$.

Remark 3.2. Let N be a proper submodule of faithful, cyclic R-module M:

1. $N \subseteq Z - rad(N)$

2. $rad(N) \subseteq Z - rad(N)$

Proof. 1. It is clear.

2. Let P be a Z-prime submodule of an R-module M containing N, since M is cyclic, faithful, then P is a prime submodule of M containing N,[5] and hence $rad(N) \subseteq P$. Also, $rad(N) \subseteq \cap P$, for all Z-prime submodule P containing N. Therefore, $rad(N) \subseteq Z - rad(N)$.

$\square$

Proposition 3.3. *Let N be a submodule of an R-module M, then $Z-rad(N) = Z - rad(Z - rad(N))$*

Proof. By remark 3.2(1) we have $Z - rad(N) \subseteq Z - rad(Z - rad(N))$. Let P be a Z-prime submodule of M containing N, then $Z - rad(N) \subseteq P$ and thus $Z - rad(Z - rad(N) \subseteq P$. Therefore $Z - rad(Z - rad(N) \subseteq \cap P$, for all Z-prime submodules P containing N. This implies that $Z - rad(Z - rad(N) \subseteq Z - rad(N)$. $\square$

Proposition 3.4. *Let N and L be submodule of an R-module M, then $Z - rad(N \cap L) \subseteq Z - rad(N) \cap Z - rad(L)$.*

Proof. Clearly $N \cap L \subseteq N$ and $N \cap L \subseteq L$. Thus $Z - rad(N \cap L) \subseteq Z - rad(N)$. By the same way $Z - rad(N \cap L) \subseteq Z - rad(L)$. It follows $Z - rad(N \cap L) \subseteq Z - rad(N) \cap Z - rad(L)$ $\square$

Proposition 3.5. *Let N and L be a submodules of an R-module M, then $Z - rad(N + L) = Z - rad(Z - rad(N) + Z + rad(L))$*

Proof. Clearly, $N \subseteq N+L$ and $L \subseteq N+L$. Hence $Z-rad(N) \subseteq Z-rad(N+L)$ and $Z-rad(L) \subseteq Z-rad(N+L)$, which implies that $Z-rad(N)-Z-rad(L) \subseteq Z-rad(N+L)$ and so $Z-rad(Z-rad(N)+Z-rad(L)) \subseteq Z-rad(Z-rad(N+L)$, but $Z - rad(Z - rad(N + L)) = Z - rad(N + L)$ by proposition 3.3, thus $Z - rad(Z - rad(N) + Z - rad(L)) \subseteq Z - rad(N + L)$.

Now, we must prove that $Z-rad(N+L) \subseteq Z-rad(Z-rad(N)+Z-rad(L))$, let P be a Z-prime submodule of M containing $Z - rad(N) + Z - rad(L)$, then P is a Z-prime submodule of M containing $N + L$ and hence $Z - rad(N + L) \subseteq P$. Also, $Z - rad(N + L) \subseteq \cap P$, for all Z-prime submodules of P containing $Z - rad(N) + Z - rad(L)$. Therefore $Z - rad(N + L) \subseteq Z - rad(Z - rad(N) + Z - rad(L))$. $\square$

Proposition 3.6. *Let N be a submodule of cylic,faithful R-module M, then $\sqrt{([N : M])}M \subseteq Z - rad(N)$ and if M is a multiplication module, then $\sqrt{([N : M)]}M = Z - rad(N)$.*

Proof. Let P be a Z-prime submodule of M containing N, then $[N : M] \subseteq [P : M]$ and hence $\sqrt{([N : M)]} \subseteq [P : M]$, since M is cylic, faithful R-module, hence $[P : M]$ is a prime ideal and hence $\sqrt{([N : M)]}M \subseteq [P : M]M \subseteq P$, thus $\sqrt{([N : M)]}M \subseteq Z - rad(N)$.

Now, if M is a multiplication module, then $\sqrt{([N : M)]}M = rad(N)$, [8] and thus $\sqrt{([N : M)]}M = Z - rad(N)$ □

In the following proposition, we give condition under which the other inclusion of proposition 3.4 holds.

Proposition 3.7. *Let N and L be submodule of an R-module M, such that whenever $N \cap L \subseteq P$, where P is a Z-prime submodule of M, we have $N \subseteq P$ or $L \subseteq P$, then $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L)$.*

Proof. If $Z - rad(N \cap L) = M$, then clearly $Z - rad(N) = Z - rad(L) = M$ and so $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L) = M$. If $Z - rad(N \cap L) \neq M$, then there exist a Z-prime submodule P, such that $N \cap L \subseteq P$. By hypothesis, $N \subseteq P$ or $L \subseteq P$ so that $Z - rad(N) \subseteq P$ or $Z - rad(L) \subseteq P$. Since this is true for all Z-prime submodule P containing $N \cap L$, thus $Z - rad(N) \cap Z - rad(L) \subseteq Z - rad(N \cap L)$ and therefore $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L)$. □

Corollary 3.8. *Let M be a cyclic, faithful R-module. Then $Z - rad(IM \cap N) = Z - rad(IM) \cap Z - rad(N)$, for every ideal I of R and every submodule N of M.*

Proof. Let N be a submodule of an R-module M and I be an ideal of R. We only have to show that if $IM \cap N \subseteq P$, where P is a Z-prime submodule of M, then either $N \subseteq P$ or $IM \subseteq P$. Now, suppose that $N \not\subseteq P$, then there exist $x \in N$ and $x \notin P$. $Ix \subseteq IM \cap N \subseteq P$, since M cyclic, faithful, then P is a prime submodule of M and $x \notin P$ thus $I \subseteq [P : M]$, therefore $IM \subseteq P$. □

Proposition 3.9. *If N and L are submodules of a cyclic, faithful R-module M, such that $\sqrt{([N : M])} + \sqrt{([L : M])} = R$, then $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L)$.*

Proof. Clearly $Z - rad(N \cap L) \subseteq Z - rad(N) \cap Z - rad(L)$, by proposition 3.4. If P is a Z-prime submodule of M containing $N \cap L$, then $[N \cap L : M] \subseteq [P : M]$, then $[N : M] \cap [L : M] \subseteq [P : M]$. Since M is cyclic, faithful,then $[P : M]$ is a prime ideal, so either $[L : M] \subseteq [P : M]$ or $[N : M] \subseteq [P : M]$. If $[N : M] \subseteq [P : M]$, then $[L : M] \not\subseteq [P : M]$ for otherwise $\sqrt{([L : M])} \subseteq [P : M]$ which is a contradiction, therefore $N \subseteq P$. This implies that $Z - rad(N) \subseteq P$ also $Z - rad(L) \subseteq P$, then $Z - rad(N) \cap Z - rad(L) \subseteq P$, thus $Z - rad(N \cap L) = Z - rad(N) \cap Z - rad(L)$. □

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